THE SURREAL NUMBERS AS A UNIVERSAL H-FIELD

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Abstract. We show that the natural embedding of the differential field of transseries into Conway's field of surreal numbers with the Berarducci-Mantova derivation is an elementary embedding. We also prove that any Hardy field embeds into the field of surreals with the Berarducci-Mantova derivation.

Introduction

Berarducci and Mantova [3, Theorem B] have recently constructed a derivation (denoted by ∂_{BM} below) on Conway's ordered field **No** of surreal numbers that makes the latter a Liouville closed H-field with constant field $\mathbb{R}$. The standard example of such an object is the ordered differential field $\mathbb{T}$ of transseries, and the question arises whether **No** with ∂_{BM} is elementarily equivalent to $\mathbb{T}$. Below we give a positive answer in a stronger form: Theorem 1. Throughout this paper we consider **No** as a differential field with derivation ∂_{BM}.

Both **No** and $\mathbb{T}$ are also exponential fields; the exponential function exp on **No** is defined in Gonshor [9]. We refer to [2, Appendix A] for the precise construction of $\mathbb{T}$, but the "generating element" x of $\mathbb{T}$ there will be denoted by $x_{\mathbb{T}}$ here, since we prefer to have x range here over arbitrary surreal numbers. It is folklore (but see Section 5 for a proof) that there is a unique embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ of ordered exponential fields with $\iota(x_{\mathbb{T}}) = \omega$ that is the identity on $\mathbb{R}$ and respects infinite sums. It follows easily from Wilkie's theorem [13] and other known facts that ι is an elementary embedding of ordered exponential fields; see Section 5 for details. The analogue for the derivation instead of the exponentiation requires more effort:

Theorem 1. *The mapping $\iota\colon \mathbb{T} \to \mathbf{No}$ is an elementary embedding of ordered differential fields.*

This answers a question posed in [3]. The main tools for proving this result come from [2, Theorems 15.0.1 and 16.0.1]. These tools enable us to reduce the proof of Theorem 1 to exhibiting **No** as a directed union of subfields $\mathbb{R}[[\omega^\Gamma]]$ that are closed under ∂_{BM} and where Γ is an ordered additive subgroup of **No** having a smallest nontrivial archimedean class; exhibiting **No** as such a directed union makes up an important part of our paper. (As a byproduct we get a new proof that $\partial_{\text{BM}}(\mathbf{No}) = \mathbf{No}$.) We use the same kind of reduction to obtain:

Theorem 2. *The surreals of countable length form a subfield of* **No** *closed under ∂_{BM}. As a differential subfield of* **No** *it is an elementary submodel of* **No**.

This also uses a result of Esterle [8] and its consequence that for any countable ordinal α, any well-ordered set of surreals of length $< \alpha$ is countable: Lemma 4.3.

Finally, we establish an embedding result for H-fields:

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Theorem 3. *Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded over $\mathbb{R}$ as an ordered differential field into* **No**.

Thus every Hardy field extending $\mathbb{R}$ embeds over $\mathbb{R}$ as an ordered differential field into **No**. Despite these excellent properties of ∂_{BM}, Schmeling's thesis [12] gives us reason to believe that ∂_{BM} is not yet the "best" derivation on **No**. We expect to address this issue in later papers.

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1. Preliminaries

Here we fix notation and terminology and summarize the results from [2, 3, 9] that we need as background material and as tools in our proofs.

Notations and terminology. Below, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$, and α, β and μ, ν range over ordinals. (The letter λ will serve another purpose, as in [3].)

As in [9], a *surreal number* is by definition a function $a\colon \mu \to \{-,+\}$ on an ordinal $\mu = \{\alpha : \ \alpha < \mu\}$. For such a we let $l(a) := \mu$ be the *length* of a. From now on we let a, b, x, y be surreal numbers. The class **No** of surreal numbers carries a canonical linear ordering $<$: $a < b$ iff a is lexicographically less than b, where by convention we set $a(\mu) := 0$ for $\mu \geqslant l(a)$ and linearly order $\{-, 0, +\}$ by $- < 0 < +$. We also have the canonical partial ordering $<_s$ on **No** given by: $a <_s b$ ("a is simpler than b") iff a is a proper initial segment of b, that is, $l(a) < l(b)$, and $a|_\mu = b|_\mu$ for $\mu := l(a)$. For sets $A, B \subseteq \mathbf{No}$ with $A < B$ (that is, $a < b$ for all $a \in A$ and $b \in B$) we let $x = A|B$ mean that x is the simplest surreal with $A < x < B$, as in [9] and [3]. We also use the terms "canonical representation" and "monomial representation" (of a surreal number) as in [3].

The ordinal α is identified with the surreal $a\colon \alpha \to \{-,+\}$ with $a(\beta) = +$ for all $\beta < \alpha$. A useful fact is the equivalence $\alpha < x \iff \alpha \dotplus 1 \leqslant_s x$, where $\alpha \dotplus 1$ is the successor ordinal to α. The subclass of **No** consisting of the ordinals is denoted by **On**. A set $S \subseteq \mathbf{No}$ is said to be *initial* if $x \in S$ whenever $x <_s y \in S$. As in [5] we set $\mathbf{No}(\alpha) = \big\{x : \ l(x) < \alpha\big\}$, an initial subset of **No**.

We refer to [9] or [3] for the inductive definitions of the binary operations of addition and multiplication on **No** that make **No** into a real closed field, with the ordinal 0 as its zero element and the ordinal 1 as its multiplicative identity. The field ordering of this real closed field is the above lexicographic linear ordering $<$. This field **No** contains $\mathbb{R}$ as an initial subfield in the way specified in [9]. The field sum $\alpha + n$ equals the ordinal sum $\alpha \dotplus n$. Each initial set $\mathbf{No}(\omega^\alpha)$ underlies an additive subgroup of **No**; see [5].

Let Γ be an (additively written) ordered abelian group. Then we set

$$\Gamma^{>} \ := \ \{\gamma \in \Gamma : \ \gamma > 0\}.$$

We use this notation also for the underlying additive groups of **No** and $\mathbb{R}$, so $\mathbf{No}^{>} = \{a : \ a > 0\}$, and $\mathbb{R}^{>} := \{r \in \mathbb{R} : \ r > 0\}$. For $\gamma \in \Gamma$ we define

$$[\gamma] \ := \ \big\{\delta \in \Gamma : \ |\delta| \leqslant n|\gamma| \text{ and } |\gamma| \leqslant n|\delta| \text{ for some } n \geqslant 1\big\},$$

the *archimedean class of* γ (in Γ). The archimedean classes of elements of Γ partition the set Γ, and we totally order this set of archimedean classes by

$$[\gamma_1] \ < \ [\gamma_2] \quad :\Longleftrightarrow \quad n|\gamma_1| \ < \ |\gamma_2| \text{ for all } n \geqslant 1.$$

Thus the least archimedean class is $[0] = \{0\}$, the *trivial* archimedean class.

The convex hull of $\mathbb{R}$ in $\mathbf{No}$ is a valuation ring V of the field $\mathbf{No}$. We consider $\mathbf{No}$ accordingly as a *valued* field whose (Krull) valuation v has V as its valuation ring. For any (Krull) valued field K with valuation v and elements $f, g \in K$ we let $f \preccurlyeq g$, $f \prec g$, $f \asymp g$, $f \sim g$ abbreviate $v(f) \geqslant v(g)$, $v(f) > v(g)$, $v(f) = v(g)$, and $v(f-g) > vf$. (See [2, Section 3.1].) We shall use these notations in particular for the valued field $\mathbf{No}$.

The omega map, the Conway normal form, and summability. We assume familiarity with Conway's omega map $x \mapsto \omega^x \colon \mathbf{No} \to \mathbf{No}^{>}$. Recall that ω^x is the simplest positive element in its archimedean class; so $\omega^x \prec \omega^y$ whenever $x < y$. See [9] for details, including the proof that each a has a unique representation

$$a \ = \ \sum_x a_x \omega^x \qquad \text{(the Conway normal form of } a\text{)}$$

with real coefficients a_x such that $E(a) := \{x : \ a_x \neq 0\}$ is a subset of $\mathbf{No}$ (not just a subclass) and is reverse well-ordered. This will be the meaning of $E(a)$ and a_x throughout. The *leading monomial of* a is ω^x with $x = \max E(a)$, for $a \neq 0$. The *terms* of a are the $a_x\omega^x$ with $a_x \neq 0$. The omega map extends the usual ordinal exponentiation $\alpha \mapsto \omega^\alpha$. Given any set $S \subseteq \mathbf{No}$ we let $\mathbb{R}[[\omega^S]]$ denote the additive subgroup of $\mathbf{No}$ consisting of the surreals a with $E(a) \subseteq S$.

Let $(a_i)_{i\in I}$ be a family of surreals; this includes I being a set. We say that (a_i) is *summable* (or that $\sum_i a_i$ exists) if $\bigcup_i E(a_i)$ is reverse well-ordered, and for each x there are only finitely many $i \in I$ with $x \in E(a_i)$; in that case we set $\sum_i a_i := \sum_x \left(\sum_i a_{i,x}\right) \omega^x$. If S is a subset of $\mathbf{No}$, then for any summable family (a_i) in $\mathbb{R}[[\omega^S]]$ we have $\sum_i a_i \in \mathbb{R}[[\omega^S]]$.

As in [3], we let $\mathfrak{M}$ denote the class of *monomials* ω^x; so $\mathfrak{M}$ is a multiplicative subgroup of $\mathbf{No}^\times$. The Conway normal form allows us to consider any surreal number a as a *generalized series*

$$a \ = \ \sum_{\mathfrak{m}\in\mathfrak{M}} a_{\mathfrak{m}}\mathfrak{m}$$

with coefficients $a_{\mathfrak{m}} \in \mathbb{R}$, monomials $\mathfrak{m} \in \mathfrak{M}$, and reverse well-ordered *support* $\operatorname{supp} a := \{\mathfrak{m} \in \mathfrak{M} : a_{\mathfrak{m}} \neq 0\} = \omega^{E(a)}$. This makes the above notion of summability for surreal numbers coincide with the corresponding notion for generalized series from [12, Section 1.5].

Next, $\mathbb{J} := \left\{a : \ E(a) \subseteq \mathbf{No}^{>}\right\}$ is the class of *purely infinite* surreals, an additive subgroup of $\mathbf{No}$ that is moreover closed under multiplication. Thus $\mathfrak{M} \cap \mathbb{J} = \mathfrak{M}^{\succ 1}$, and $\mathbf{No} = \mathbb{J} \oplus \mathbb{R} \oplus \mathbf{No}^{\prec 1}$.

Exponentiation, and the functions g and h. Gonshor [9] gave an inductive definition of the exponential function $\exp\colon \mathbf{No} \to \mathbf{No}^{>}$, and established its basic properties. These include exp being an order-preserving isomorphism from the additive group of $\mathbf{No}$ onto its multiplicative group of positive elements. The inverse of exp is of course denoted by $\log\colon \mathbf{No}^{>} \to \mathbf{No}$. The nth iterate of the map $\exp\colon \mathbf{No} \to \mathbf{No}$ is denoted by $\exp_n$, so $\exp_0$ is the identity map on $\mathbf{No}$, and

$\exp_1(x) = \exp(x)$. Also $\mathrm{e}^x := \exp(x)$. The logarithmic map log maps $\mathbf{No}^{>\mathbb{N}}$ into itself; the nth iterate of the restriction of log to a map $\mathbf{No}^{>\mathbb{N}} \to \mathbf{No}^{>\mathbb{N}}$ is denoted by $\log_n$, so $\log_0$ is the identity map on $\mathbf{No}^{>\mathbb{N}}$ and $\log_1(x) = \log(x)$ for $x > \mathbb{N}$.

The exponential map exp and the omega-map $x \mapsto \omega^x$ are related by the order preserving bijection $g\colon \mathbf{No}^{>} \to \mathbf{No}$, which satisfies

$$\exp(\omega^x) \ = \ \omega^{\omega^{g(x)}} \qquad \text{for all } x > 0.$$

We have $g(n) = n$ for all n. More generally, Theorem 10.14 in [9] says that $g(\alpha) = \alpha$ unless $\varepsilon \leqslant \alpha < \varepsilon + \omega$ for some ε-number, in which case $g(\alpha) = \alpha + 1$. (An *ε-number* is an ordinal ε such that $\omega^\varepsilon = \varepsilon$.) We shall need $g(x)$ mainly in the other extreme case where x has the form $\omega^{-\alpha}$. Here Theorem 10.15 in [9] gives $g(\omega^{-\alpha}) = -\alpha + 1$.

We also use the inverse $h\colon \mathbf{No} \to \mathbf{No}^{>}$ of g. Note that

$$\omega^{\omega^y} \ = \ \exp(\omega^{h(y)}) \qquad \text{for all } y.$$

The result above for $g(\omega^{-\alpha})$ yields $h(-\alpha + 1) = \omega^{-\alpha}$, from which we get

$$\log \omega^{\omega^{-\alpha+1}} \ = \ \omega^{\omega^{-\alpha}}.$$

Applying this to the ordinal $\alpha + 1$ instead of α we get

$$\log \omega^{\omega^{-\alpha}} \ = \ \omega^{\omega^{-(\alpha+1)}}.$$

From [9] we have $\exp(\mathbb{J}) = \mathfrak{M}$. Thus besides the Conway normal form and the series representation, any surreal number a also has a unique representation

$$a \ = \ \sum_{j \in \mathbb{J}} a_j \,\mathrm{e}^j \qquad \text{(exponential normal form of } a\text{)}$$

with real coefficients a_j and reverse well-ordered $\{j \in \mathbb{J} : a_j \neq 0\}$; this is also called the *Ressayre form of* a. For nonzero a with leading monomial e^b, $b \in \mathbb{J}$, we set $\ell(a) := b$. Then $-\ell\colon \mathbf{No}^{\times} \to \mathbb{J}$ is a (Krull) valuation on the field $\mathbf{No}$, and

$$\big\{a : -\ell(a) \geqslant 0\big\} \ = \ \big\{a : \ |a| \leqslant r \text{ for some } r \in \mathbb{R}^{\geqslant 0}\big\} \ = \ V,$$

so we may consider $-\ell$ as the valuation of our valued field $\mathbf{No}$. Important in [3] is also the class $\mathfrak{A}$ of *log-atomic* surreals, consisting of the $a > \mathbb{N}$ all whose iterated logarithms $\log_n a$ lie in $\mathfrak{M}$. We have $\mathfrak{A} \subseteq \mathfrak{M}^{\succ 1}$ and $\exp(\mathfrak{A}) = \log(\mathfrak{A}) = \mathfrak{A}$. It follows from $\mathfrak{A} \subseteq \mathfrak{M}$ that if $x, y \in \mathfrak{A}$ and $x < y$, then $x \prec y$. (In [3] the class of log-atomic surreals is denoted by $\mathbb{L}$, but this notation conflicts with ours in other papers.)

Surreal derivations. We summarize here some results from [3] as needed, and add a few remarks. A *surreal derivation* is a derivation ∂ on the field $\mathbf{No}$ such that

- (SD1) $\big\{a : \partial(a) = 0\big\} = \mathbb{R}$;
- (SD2) $\partial(a) > 0$ for all $a > \mathbb{R}$;
- (SD3) $\partial\big(\exp(a)\big) = \partial(a)\exp(a)$ for all a;
- (SD4) for any summable family (a_i) of surreals, the family $\big(\partial(a_i)\big)$ is also summable, and $\partial\left(\sum_i a_i\right) = \sum_i \partial(a_i)$.

The ordered field $\mathbf{No}$ equipped with any surreal derivation is an H-field; this doesn't need (SD3) or (SD4). The particular derivation ∂_{BM} is surreal, maps $\mathfrak{A}$ into $\mathfrak{M}$, and is obtained in [3] as a special case of a rather general construction. Before we get to that, we mention Proposition 6.5 and Theorem 6.32 from that paper:

(BM1) If ∂ is a surreal derivation, then for all $x, y > \mathbb{N}$ with $x - y > \mathbb{N}$ we have

$$\log \partial(x) - \log \partial(y) \ \prec\ x - y.$$

(BM2) Any map $D\colon \mathfrak{A} \to \mathbb{R}^{>}\mathfrak{M}$ such that for all $x, y \in \mathfrak{A}$,

$$D(\exp x) \ =\ D(x) \exp x, \quad \log D(x) - \log D(y) \ \prec\ \max(x, y),$$

extends to a surreal derivation.

Thus (BM2) is a partial converse to (BM1), although the condition in (BM2) that D takes only values in $\mathbb{R}^{>}\mathfrak{M}$ seems a rather severe restriction. We define a *pre-derivation* to be a map $D\colon \mathfrak{A} \to \mathbb{R}^{>}\mathfrak{M}$ as in (BM2). Note that if D is a pre-derivation, then

$$D(a) = \left(\prod_{m<n} \log_m a \right) \cdot D(\log_n a) \qquad \text{for all } a \in \mathfrak{A} \text{ and all } n. \tag{$*$}$$

A pre-derivation D actually extends canonically to a surreal derivation ∂_D. To define ∂_D in terms of D we rely on the notion of *path derivatives*, introduced in [10], further developed in [12], and adapted to the surreal setting in [3]. A *path* is a function $P\colon \mathbb{N} \to \mathbb{R}^{\times}\mathfrak{M}$ such that $P(n+1)$ is a term of $\ell(P(n))$, for all n. Given x, the paths P such that $P(0)$ is a term of x are the elements of a set $\mathcal{P}(x)$. For $x \in \mathfrak{A}$ there is a unique path $P \in \mathcal{P}(x)$; it is given by $P(n) = \log_n x$. Thus if P is a path and $P(m) \in \mathfrak{A}$, then $P(n) = \log_{n-m} P(m)$ for all $n \geqslant m$, so $P(n) \in \mathfrak{A}$ for all $n \geqslant m$.

Let D be a pre-derivation. The *path derivative* $\partial_D(P) \in \mathbb{R}\mathfrak{M}$ for a path P is defined as follows, with $(*)$ guaranteeing independence of n in (1):

(1) if $P(n) \in \mathfrak{A}$, then $\partial_D(P) := \left(\prod_{m<n} P(m)\right) \cdot D(P(n))$;
(2) if $P(n) \notin \mathfrak{A}$ for all n, then $\partial_D(P) := 0$.

The rationale behind path derivatives is the following proposition:

(BM3) For each a the family $\left(\partial_D(P)\right)_{P \in \mathcal{P}(a)}$ is summable.

This result is stated in [3, Proposition 6.20] only for one particular pre-derivation, but, as the authors mention, the proof extends to any pre-derivation. In view of (BM3) we can now define $\partial_D\colon \mathbf{No} \to \mathbf{No}$ by

$$\partial_D(a) \ :=\ \sum_{P \in \mathcal{P}(a)} \partial_D(P).$$

It follows from $(*)$ that ∂_D extends D, and the arguments in [3, Section 6] show that ∂_D is a surreal derivation.

Results from [2]**.** To state the relevant facts, we recall from [1] or [2] that an *H-field* is by definition an ordered differential field K with derivation ∂ and constant field $C = \left\{f \in K :\ \partial(f) = 0\right\}$ such that:

(H1) $\partial(f) > 0$ for all $f \in K$ with $f > C$;
(H2) $\mathcal{O} = C + \smallo$, where $\mathcal{O}$ is the convex hull of C in K, and $\smallo$ is the maximal ideal of the valuation ring $\mathcal{O}$.

Let K be an H-field, and let $\mathcal{O}$ and $\smallo$ be as in (H2). Thus K is a valued field with valuation ring $\mathcal{O}$. We consider K in the natural way as an $\mathcal{L}$-structure, where

$$\mathcal{L} := \{\, 0,\, 1,\, +,\, -,\, \times,\, \partial,\, \leqslant,\, \preccurlyeq \,\}$$

is the language of ordered valued differential fields; in particular,

$$f \preccurlyeq g \iff f \in \mathcal{O}g \iff |f| \leqslant c|g| \text{ for some } c \geqslant 0 \text{ in } C.$$

Given $f \in K$ we also write f' instead of $\partial(f)$, and we set $f^\dagger := f'/f$ for $f \neq 0$, so $(fg)^\dagger = f^\dagger + g^\dagger$ and $(1/f)^\dagger = -f^\dagger$ for $f, g \in K^\times$. A useful subset of the value group $\Gamma := v(K^\times)$ of the valued field K is

$$\Psi := \Psi_K := \big\{v(f^\dagger) : f \in K^\times,\ f \not\asymp 1\big\} = \big\{v(f^\dagger) : f \in K,\ f > C\big\}.$$

As in [2] we call K *grounded* if Ψ has a largest element. For the convenience of the reader we include a proof of the following wellknown fact.

Lemma 1.1. *Assume K has constant field $C = \mathbb{R}$. Then K is grounded iff Γ has a smallest nontrivial archimedean class.*

Proof. Let $f, g \in K$, $f, g > C$. Suppose the archimedean class $\big[v(f)\big] = \big[v(1/f)\big]$ of $v(f)$ is greater than $\big[v(g)\big]$. This means: $v(f) < nv(g) = v(g^n) < 0$ for all $n \geqslant 1$. Hence $f^\dagger > (g^n)^\dagger = ng^\dagger > 0$ for all $n \geqslant 1$, by [1, Lemma 1.4], so $v(f^\dagger) < v(g^\dagger)$. A similar argument (which doesn't need $C = \mathbb{R}$) shows that if $\big[v(f)\big] = \big[v(g)\big]$, then $v(f^\dagger) = v(g^\dagger)$. Thus we have an order-reversing bijection $\big[v(f)\big] \mapsto v(f^\dagger)$ ($f \in K$, $f > C$) from the set of nontrivial archimedean classes of Γ onto Ψ. □

An *H-subfield of K* is by definition an ordered differential subfield of K that is an H-field. In [2] we axiomatized the elementary (= first-order) theory of the H-field $\mathbb{T}$ of transseries. This (complete) theory is called $T^{\mathrm{nl}}_{\mathrm{small}}$ there and its models are exactly the H-fields K satisfying the following (first-order) conditions:

(1) the derivation of K is small, that is, $\partial\smallo \subseteq \smallo$;
(2) K is Liouville closed;
(3) K is $\upomega$-free;
(4) K is newtonian.

(An H-field K is said to be *Liouville closed* if it is real closed and for all $f \in K$ there exists $g \in K$ with $g' = f$ and an $h \in K^\times$ such that $h^\dagger = f$; for the definition of "$\upomega$-free" and "newtonian" we refer to the Introduction of [2].) Dropping the smallness axiom (1), we get the incomplete but model complete theory T^{nl}; see [2, Chapter 16]. The H-field $\mathbb{T}$ satisfies (3) and (4) by [2, Corollary 11.7.15 and Theorem 15.0.1], which for an arbitrary H-field K amount to the following:

If $\partial K = K$ and K is a directed union of spherically complete grounded H-subfields, then K is $\upomega$-free and newtonian.

The condition $\partial K = K$ is automatically satisfied if K is a directed union of spherically complete grounded H-subfields E such that for some $\phi \in E$ we have $v(\phi) = \max \Psi_E$ and $\phi \in \partial K$, by [2, Corollary 15.2.4].

2. Infinite Products and Log-atomic Surreals

The pre-derivation D in [3] with $\partial_D = \partial_{\mathrm{BM}}$ is defined by a certain identity. Towards the end of this section we give this identity a more suggestive form, which we found useful. But we begin with some remarks on ε-numbers, which play an important role in the next sections.

Remarks on ε-numbers. Throughout this paper ε will denote an ε-number, that is, ε is an ordinal such that $\omega^{\varepsilon} = \varepsilon$.

Lemma 2.1. *For any α there is a least ε-number $\varepsilon(\alpha) \geqslant \alpha$. Moreover, if α is infinite, then* $\operatorname{card}(\varepsilon(\alpha)) = \operatorname{card}(\alpha)$.

Proof. The recursion defining ω^{α} as a function of α easily yields that this function is strictly increasing, with $\omega^{\alpha} \geqslant \alpha$, $\operatorname{card}(\omega^{\alpha}) = \max\big(\aleph_0, \operatorname{card}(\alpha)\big)$, and thus $\operatorname{card}(\omega^{\alpha}) = \operatorname{card}(\alpha)$ if α is infinite. Now define α_n as a function of n by the recursion $\alpha_0 = \alpha$ and $\alpha_{n+1} = \omega^{\alpha_n}$. Then $\sup_n \alpha_n$ is clearly the least ε-number $\geqslant \alpha$, and it has the same cardinality as α if the latter is infinite. $\square$

If κ is an uncountable cardinal, then by the remarks in the proof above we have $\omega^{\alpha} < \kappa$ for all $\alpha < \kappa$. Thus uncountable cardinals are ε-numbers. The least ε-number is denoted by ε_0, as usual, so $\varepsilon_0 = \sup_n \omega_n$ where the ω_n are defined by the recursion $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$.

Infinite products of monomials. Recall that $\mathfrak{M}$ is the multiplicative group of monomials ω^{a}. For a family $(\mathfrak{m}_i)$ in $\mathfrak{M}$ we say that $\prod_i \mathfrak{m}_i$ exists if $\sum_i a_i$ exists, with $\mathfrak{m}_i = \omega^{a_i}$ for all i, and in that case, we set

$$\prod_i \mathfrak{m}_i \ := \ \omega^{\sum_i a_i} \in \mathfrak{M}.$$

The rules for manipulating these infinite products are easy consequences of those for infinite sums, and we shall freely use them below. Note in particular that if $(\mathfrak{m}_i)$ is a family in $\mathfrak{M}$ and $\prod_i \mathfrak{m}_i$ exists, then $\prod_i \mathfrak{m}_i^{-1}$ exists and equals $(\prod_i \mathfrak{m}_i)^{-1}$.

In our definition of infinite products we could have represented monomials as exponentials of elements in $\mathbb{J}$ instead of as powers of ω. Indeed, the equivalence between these options follows from the next two lemmas:

Lemma 2.2. *Let (a_i) be a summable family in $\mathbb{J}$. Then $\prod_i \exp(a_i)$ exists, and*

$$\exp\left(\sum_i a_i\right) \ = \ \prod_i \exp(a_i).$$

Proof. We have $a_i = \sum_{x>0} a_{i,x}\omega^{x}$, so by [9, Theorem 10.13],

$$\exp(a_i) \ = \ \omega^{b_i}, \quad b_i \ := \ \sum_{x>0} a_{i,x}\omega^{g(x)},$$

so $E(b_i) = g(E(a_i))$. Since $\sum_i a_i$ exists, so does $\sum_i b_i$, and hence $\prod_i \exp(a_i) = \prod_i \omega^{b_i}$ exists, and $\prod_i \exp(a_i) = \omega^{\sum_i b_i}$. Moreover, with $\sum_i a_i = \sum_{x>0} a_x\omega^{x}$, we have $\sum_i b_i = \sum_{x>0} a_x\omega^{g(x)}$. Hence again by [9, Theorem 10.13],

$$\prod_i \exp(a_i) \ = \ \omega^{\sum_{x>0} a_x\omega^{g(x)}} \ = \ \exp\left(\sum_{x>0} a_x\omega^{x}\right) \ = \ \exp\left(\sum_i a_i\right),$$

as claimed. $\square$

Lemma 2.3. *Let $(\mathfrak{m}_i)$ be a family in $\mathfrak{M}$ such that $\prod_i \mathfrak{m}_i$ exists. Then $\sum_i \log \mathfrak{m}_i$ exists, and* $\log \prod_i \mathfrak{m}_i = \sum_i \log \mathfrak{m}_i$.

Proof. We have $\mathfrak{m}_i = \exp(a_i)$ with $a_i \in \mathbb{J}$, so $a_i = \sum_{x>0} a_{i,x}\omega^x$, hence

$$\mathfrak{m}_i \ = \ \omega^{b_i}, \qquad b_i \ := \ \sum_{x>0} a_{i,x}\omega^{g(x)}$$

by [9, Theorem 10.13]. Since the product $\prod_i \mathfrak{m}_i$ exists, so does $\sum_i b_i$, and therefore $\sum_i a_i = \sum_i \log \mathfrak{m}_i$ exists. Moreover, and again by [9, Theorem 10.13],

$$\prod_i \mathfrak{m}_i \ = \ \omega^{\sum_i b_i} \ = \ \omega^{\sum_{x>0} a_x \omega^{g(x)}} \ = \ \exp\left(\sum_{x>0} a_x\omega^x\right), \quad a_x := \sum_i a_{i,x},$$

and so $\log \prod_i \mathfrak{m}_i = \sum_{x>0} a_x\omega^x = \sum_i a_i$. □

Log-atomic surreals. Recall that $\mathfrak{A} \subseteq \mathfrak{M}^{\succ 1}$ is the class of log-atomic surreals. See [3, Sections 1, 5] for the order-preserving bijection $x \mapsto \lambda_x \colon \mathbf{No} \to \mathfrak{A}$ and for the fact that $\lambda_x \leqslant_s \lambda_y$ iff $x \leqslant_s y$. It follows from $\exp(\omega^x) = \omega^{\omega^{g(x)}}$ that $\mathfrak{A} \subseteq \omega^{\mathfrak{M}}$. Thus for any well-ordered index set I and strictly decreasing map $i \mapsto \lambda_i \colon I \to \mathfrak{A}$ the product $\prod_i \lambda_i$ exists. We shall use Proposition 2.6 and Corollary 2.9 below to define the pre-derivation $\partial_{\mathrm{BM}}|_{\mathfrak{A}}$.

Lemma 2.4. *Let $\mathfrak{m} = A|B$ be a monomial representation with $\mathfrak{m} \succ 1$. Then*

$$\exp(\mathfrak{m}) \ = \ \big(\mathfrak{m}^{\mathbb{N}} \cup \exp(A)\big)\,\big|\,\exp(B).$$

Proof. For $\mathfrak{m}' < \mathfrak{m}$ with $\mathfrak{m}' <_s \mathfrak{m}$ we have $\mathfrak{m}' \leqslant a$ for some $a \in A$ (since $A < \mathfrak{m}' < \mathfrak{m} < B$ gives $\mathfrak{m} \leqslant_s \mathfrak{m}'$). Likewise, for $\mathfrak{m} < \mathfrak{m}'' <_s \mathfrak{m}$, we have $b \leqslant \mathfrak{m}''$ for some $b \in B$. It follows that for $\mathfrak{m}'$ as above and $k \in \mathbb{N}^{\geqslant 1}$ we have $\exp(\mathfrak{m}')^k \leqslant \exp(a)$ for some $a \in A$, and that for $\mathfrak{m}''$ as above and $k \in \mathbb{N}^{\geqslant 1}$ we have $\exp(b) \leqslant \exp(\mathfrak{m}'')^{1/k}$ for some $b \in B$. This yields the desired result in view of [3, Theorem 3.8 (1)]. □

The monomial representation $\omega = \mathbb{N}|\emptyset$ shows that in the conclusion of Lemma 2.4 we cannot drop $\mathfrak{m}^{\mathbb{N}}$. Below we use the binary relations $\asymp^L$ and $\prec^L$ from [3]. Let $x = \{x'\}|\{x''\}$ be the canonical representation of x, and let j, k range over $\mathbb{N}^{\geqslant 1}$. Then by [3, Definition 5.12], the defining representation of λ_x is given by

$$\lambda_x \ = \ \big\{k, \exp_j\big(k \log_j(\lambda_{x'})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_j(\lambda_{x''})\big)\big\}\,.$$

Proposition 2.5. *We have $\lambda_{x+1} = \exp(\lambda_x)$, and thus $\lambda_{x-1} = \log(\lambda_x)$.*

Proof. Let $x = \{x'\}|\{x''\}$ be the canonical representation of x. Then $1 = 0|\emptyset$ gives $x+1 = \{x, x'+1\}|\{x''+1\}$. Assume inductively that $\lambda_{x'+1} = \exp(\lambda_{x'})$ and $\lambda_{x''+1} = \exp(\lambda_{x''})$ for all x' and x''. With j, k ranging over $\mathbb{N}^{\geqslant 1}$, [3, 5.15] gives

$$\begin{aligned} \lambda_{x+1} \ &= \ \big\{k, \exp_j\big(k\log_j(\lambda_x)\big), \exp_j\big(k\log_j(\lambda_{x'+1})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_j(\lambda_{x''+1})\big)\big\} \\ &= \ \big\{k, \exp_j\big(k\log_j(\lambda_x)\big), \exp_j\big(k\log_{j-1}(\lambda_{x'})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_{j-1}(\lambda_{x''})\big)\big\}\,. \end{aligned}$$

The defining representation $\lambda_x = A|B$ is monomial, and the above gives $\lambda_{x+1} = \mathbb{N} \cup S \cup \exp(A)|\exp(B)$ where S includes $\lambda_x^{\mathbb{N}}$ and all elements of S are $\asymp^L \lambda_x$. Since $\lambda_x \prec^L \exp(\lambda_x)$, it follows easily from Lemma 2.4 that $\lambda_{x+1} = \exp(\lambda_x)$. □

Lemma 2.6. *We have $\lambda_{-\alpha} = \omega^{\omega^{-\alpha}}$.*

Proof. By induction on α. The case $\alpha = 0$ holds since $\lambda_0 = \omega$. Assuming it holds for a certain α, we have

$$\lambda_{-(\alpha+1)} \ = \ \log\lambda_{-\alpha} \ = \ \log\omega^{\omega^{-\alpha}} \ = \ \omega^{\omega^{-(\alpha+1)}}.$$

Next, let μ be an infinite limit ordinal. Then $-\mu = \emptyset|\{-\alpha :\ \alpha < \mu\}$, and so by [3, 5.15] and with j, k ranging over $\mathbb{N}^{\geqslant 1}$ we have

$$\lambda_{-\mu} \ =\ \mathbb{N}\,\big|\,\big\{\exp_j\big(\tfrac{1}{k}\log_j \lambda_{-\alpha}\big)\big\}\,.$$

Now $\exp_j\big(\frac{1}{k}\log_j \lambda_{-\alpha}\big) \asymp^L \lambda_{-\alpha} \succ^L \lambda_{-\beta}$ when $\alpha < \beta$, so by cofinality and the inductive assumption we have

$$\lambda_{-\mu} \ =\ \mathbb{N}\,\big|\,\big\{\omega^{\omega^{-\alpha}} :\ \alpha < \mu\big\}.$$

From $\mathbb{N} < \omega^{\omega^{-\mu}} < \omega^{\omega^{-\alpha}}$ for all $\alpha < \mu$, we get $\lambda_{-\mu} \leqslant_s \omega^{\omega^{-\mu}}$. Take a such that $\lambda_{-\mu} = \omega^{\omega^{-a}}$. Then $\lambda_{-\mu} < \omega^{\omega^{-\alpha}}$ for $\alpha < \mu$ gives $\omega^{-a} < \omega^{-\alpha}$ for all $\alpha < \mu$, and thus $a > \alpha$ for all $\alpha < \mu$. This yields $\mu \leqslant_s a$, and thus $\omega^{\omega^{-\mu}} \leqslant_s \lambda_{-\mu}$, hence $a = \mu$. $\square$

Lemma 2.7. *For $\lambda \in \mathfrak{A}$ we have: $\lambda \ <\ \lambda_{-\alpha} \iff \lambda_{-(\alpha+1)} \ \leqslant_s\ \lambda$.*

Proof. For $\lambda = \lambda_x$ we have the equivalences

$$\begin{aligned}\lambda_x \ <\ \lambda_{-\alpha} \iff x \ <\ -\alpha \iff \alpha \ <\ -x \iff \alpha+1 \ \leqslant_s\ -x\\ \iff -(\alpha+1) \ \leqslant_s\ x \iff \lambda_{-(\alpha+1)} \ \leqslant_s\ \lambda_x.\end{aligned}$$ $\square$

Transfinitely iterating the logarithm function. In view of $\lambda_{-n} = \log_n \omega$ and the proof of Lemma 2.6 it is suggestive to think of $\lambda_{-\alpha}$ as the α times iterated function log evaluated at ω. Accordingly we set $\log_\alpha \omega := \lambda_{-\alpha}$. We note that for $\beta < \alpha$ we have $-\beta <_s -\alpha$, so $\omega^{-\beta} <_s \omega^{-\alpha}$, and thus $\log_\beta \omega <_s \log_\alpha \omega$.

Lemma 2.8. *Suppose α is an infinite limit ordinal. Then $\log_\alpha \omega$ is the simplest surreal $x > \mathbb{N}$ such that $x < \log_\beta \omega$ for all $\beta < \alpha$.*

Proof. First, $\mathbb{N} < \log_\alpha \omega < \log_\beta \omega$ for all $\beta < \alpha$. Let x be the simplest surreal $> \mathbb{N}$ such that $x < \log_\beta \omega$ for all $\beta < \alpha$. Then x is the simplest positive element in its archimedean class, so $x = \omega^y$ with $y > 0$. Then $x = \omega^y < \omega^{\omega^{-\beta}}$ for $\beta < \alpha$ gives $y < \omega^{-\beta}$ for all $\beta < \alpha$. Then y is the simplest positive element in its archimedean class: if $0 < y_0 \leqslant_s y$ and $y_0 \leqslant ny$, then $\omega^{y_0} \leqslant_s \omega^y = x$ and $\mathbb{N} < \omega^{y_0} \leqslant x^n < \log_\beta \omega$ for all $\beta < \alpha$, so $\omega^{y_0} = \omega^y$, and thus $y_0 = y$. Hence $y = \omega^z$ with $z < -\beta$ for all $\beta < \alpha$, and thus $z \leqslant -\alpha \leqslant_s z$. Therefore, $\omega^{-\alpha} \leqslant_s \omega^z = y$, so

$$\log_\alpha \omega \ =\ \omega^{\omega^{-\alpha}} \ \leqslant_s\ \omega^y \ =\ x,$$

and thus $\log_\alpha \omega = x$. $\square$

The surreals $\log_\alpha \omega$ occur in the definition of ∂_{BM} later in this section.

The κ-numbers. The definition of ∂_{BM} in [3] also involves the surreals $\kappa_x \in \mathfrak{A}$ defined by Kuhlmann and Matusinski [11]. This is only needed for $x = -\alpha$, and it follows from the results in [11] that $\kappa_{-\alpha} = \omega^{\omega^{-\omega\alpha}}$, where $\omega\alpha$ is the usual ordinal product. Thus in view of Lemma 2.6:

Corollary 2.9. *We have $\kappa_{-\alpha} \ =\ \lambda_{-\omega\alpha} \ =\ \omega^{\omega^{-\omega\alpha}} \ =\ \log_{\omega\alpha} \omega$.*

We also use the binary relations $\preccurlyeq^K$, $\succ^K$, and $\asymp^K$ on $\mathbf{No}^{>\mathbb{N}}$ defined by

$$\begin{aligned} x \ \preccurlyeq^K\ y \quad &\iff \quad x \leqslant \exp_n(y) \text{ for some } n,\\ x \ \succ^K\ y \quad &\iff \quad x > \exp_n(y) \text{ for all } n,\\ x \ \asymp^K\ y \quad &\iff \quad x \preccurlyeq^K y \text{ and } y \preccurlyeq^K x. \end{aligned}$$

We refer to [3, 5.3] for proofs of some basic facts about these relations and the κ_x such as: $\asymp^K$ is an equivalence relation on $\mathbf{No}^{>\mathbb{N}}$ with convex equivalence classes, every $\asymp^K$-equivalence class has a unique element κ_x in it, and this element is the simplest element of this equivalence class. Also, $\kappa_x \leqslant_s \kappa_y$ iff $x \leqslant_s y$.

Defining the pre-derivation for ∂_{BM}. The pre-derivation D with $\partial_D = \partial_{\mathrm{BM}}$ is denoted by $\partial_{\mathbb{L}}$ in [3, Definition 6.7], and by $\partial_{\mathfrak{A}}$ in this paper. It is given by

$$\partial_{\mathfrak{A}}(\lambda) \ := \ \prod_n \log_n \lambda \Big/ \prod_\alpha \log_\alpha \omega$$

with α in the denominator ranging over the ordinals such that $\log_\alpha \omega \geqslant \log_n \lambda$ for some n; to facilitate comparison with [3] we note that this condition on α is equivalent to $\lambda \preccurlyeq^K \log_\alpha \omega$. (The products on the right exist, since $\log_n \lambda$ and $\log_\alpha \omega$ are strictly decreasing as functions of n and α, respectively.) The above defining identity for $\partial_{\mathfrak{A}}$ simplifies the expression in [3] by our use of infinite products (instead of exponentials of infinite sums), and of Lemma 2.6 and Corollary 2.9 (to get rid of κ-numbers). As [3, Section 9] shows, $\partial_{\mathfrak{A}}$ is in a certain technical sense the *simplest* pre-derivation.

If $\lambda > \exp_n \omega$ for all n, then $\partial_{\mathfrak{A}}(\lambda) = \prod_n \log_n \lambda$. Another special case is $\partial_{\mathfrak{A}}(\log_\alpha \omega) = 1/\prod_{\beta<\alpha} \log_\beta \omega$, in particular, $\partial_{\mathfrak{A}}(\omega) = 1$. For ε-numbers we get the following (not needed later, but included as an example):

Lemma 2.10. *We have* $\log_n \varepsilon = \omega^{\omega^{\varepsilon - n}}$. *Hence* $\varepsilon \in \mathfrak{A}$ *and*

$$\partial_{\mathfrak{A}}(\varepsilon) \ = \ \omega^{\omega^{\varepsilon}+\omega^{\varepsilon-1}+\omega^{\varepsilon-2}+\cdots} \ = \ \omega^{\varepsilon/(1-\omega^{-1})}.$$

Proof. From [9, pp. 179, 180] we get that if b, as a sequence of pluses and minuses, equals ε followed by $\varepsilon\omega n$ minuses, with $n \geqslant 1$ and $\varepsilon\omega n$ being the ordinal product, then $b = \omega^{\varepsilon-n}$, and $g(b) = \varepsilon - (n-1)$. In other words,

$$g\big(\omega^{\varepsilon-n}\big) \ = \ \varepsilon - (n-1) \qquad (n \geqslant 1).$$

Using this we prove the lemma by induction on n. The case $n = 0$ is clear. Assume inductively that $\log_n \varepsilon = \omega^{\omega^{\varepsilon-n}}$. Since $g\big(\omega^{\varepsilon-(n+1)}\big) = \varepsilon - n$, this gives

$$\exp\big(\omega^{\omega^{\varepsilon-(n+1)}}\big) \ = \ \omega^{\omega^{\varepsilon-n}},$$

from which we get $\log_{n+1} \varepsilon = \omega^{\omega^{\varepsilon-(n+1)}}$, as desired. □

3. Exhibiting No as a Suitable Directed Union

At the end of Section 1 we explained how proving $\mathbb{T} \equiv \mathbf{No}$ (as differential fields) reduces to representing $\mathbf{No}$ as a directed union of spherically complete grounded H-subfields. In this section we obtain such a representation. The reader should beware of considering $\mathbf{No}$ itself as spherically complete, even though the Conway normal form is sometimes summarized as "$\mathbf{No} = \mathbb{R}(\!(\omega^{\mathbf{No}})\!)$". This is misleading, however, since it suggests that a series like $\sum_\alpha \omega^{-\alpha}$, where the sum is over all ordinals α, is a surreal number. It might perhaps be viewed as a surreal number in a strictly larger set-theoretic universe, but not in the one we are (tacitly) working in. A better way of understanding $\mathbf{No}$ as a valued field is as the directed union $\bigcup_\Gamma \mathbb{R}[[\omega^\Gamma]]$ with Γ ranging over the subsets of $\mathbf{No}$ that underly an additive subgroup of $\mathbf{No}$; for example, any α gives $\mathbf{No}(\omega^\alpha)$ as such a Γ. For any such Γ the corresponding $\mathbb{R}[[\omega^\Gamma]]$

is indeed a spherically complete valued subfield of $\mathbf{No}$, but in general $\mathbb{R}[[\omega^\Gamma]]$ is not closed under ∂_{BM}, and even if it is, it might not be grounded.

In this section we show that for $S = \mathbf{No}(\varepsilon) \cup \{-\varepsilon\}$, with ε any ε-number, the Hahn subgroup $\Gamma = \mathbb{R}[[\omega^S]]$ of $\mathbf{No}$ gives rise to a spherically complete valued subfield $\mathbb{R}[[\omega^\Gamma]]$ that is closed under ∂_{BM} and grounded as an H-subfield of $\mathbf{No}$.

A length bound for h. This very useful bound is as follows:

Lemma 3.1. $l\big(h(y)\big) \leqslant \omega^{l(y)+1}$.

Proof. By [9, p. 172] the canonical representation $y = \{y'\}|\{y''\}$ yields

$$h(y) = \big\{0, h(y')\big\}\big|\big\{h(y''), \omega^y/2^n\big\}.$$

We can assume inductively that the lemma holds for the y' and y'' instead of y, and thus $l\big(h(y')\big) \leqslant \omega^{l(y')+1} < \omega^{l(y)+1}$ for all y', and likewise with y'' instead of y'. Also, $l(\omega^y/2^n) \leqslant l(\omega^y)l(1/2^n) < \omega^{l(y)}\omega = \omega^{l(y)+1}$, using [5, Lemmas 3.6 and 4.1]. Now appeal to [9, Theorem 2.3]. □

Recall from Section 1 that $h(-\alpha) = \omega^{-(\alpha+1)}$, and so $h(0) = \omega^{-1}$ shows that for $y = 0$ the upper bound in Lemma 3.1 is attained.

Some spherically complete initial subfields of No. In this subsection we fix an initial subset I of $\mathbf{No}$. Then $\Gamma := \mathbb{R}[[\omega^I]]$ is an initial additive subgroup of $\mathbf{No}$ by the proof of Theorem 18 in [7]. (That theorem considers Hahn fields rather than the Hahn group Γ, but the same ideas work; we stress that it is the proof of that theorem rather than its statement that matters here.) Moreover, as Philip Ehrlich mentioned to one of us:

Lemma 3.2. *Suppose I has a least element a. Then $a = -\alpha$ for some α, and Γ has a least nontrivial archimedean class represented by ω^a.*

Proof. Taking the longest initial segment of a consisting of minus signs we get the largest ordinal α with $-\alpha \leqslant_s a$. Then $-\alpha \in I$ and $-\alpha \leqslant a$, so $-\alpha = a$. □

Since Γ is initial and an ordered additive group it leads to the initial subfield $K := \mathbb{R}[[\omega^\Gamma]]$ of $\mathbf{No}$. Note that K is spherically complete, and if (a_i) is a family in K for which $\sum_i a_i$ exists, then $\sum_i a_i \in K$. Now $\Gamma = \mathbb{R}[[\omega^I]]$ is also closed under infinite sums, so if $(\mathfrak{m}_i)$ is a family in $\mathfrak{M} \cap K$ such that $\prod_i \mathfrak{m}_i$ exists, then $\prod_i \mathfrak{m}_i \in K$. Thus K is closed under infinite sums, and also under infinite products of monomials. This is very useful in showing that for suitable choices of I the field K is closed under certain surreal derivations. Note however, that if I has a least element, then $K^{>\mathbb{N}}$ is not closed under log: if $-\alpha$ is the least element of I, then $\log_\alpha \omega = \omega^{\omega^{-\alpha}} \in K$, but $\log_{\alpha+1} \omega \notin K$, as $-(\alpha+1) \notin I$.

In order to discuss examples we set $a^r := \exp(r \log a)$ for $a > 0$ and $r \in \mathbb{R}$, and note agreement with the previously defined ω^r when $a = \omega$. Moreover,

$$(\log_\alpha \omega)^r = \omega^{r\omega^{-\alpha}} \qquad (r \in \mathbb{R}),$$

by the definition of a^r, using also $g\big(\omega^{-(\alpha+1)}\big) = -\alpha$ and [9, Theorem 10.13].

Examples. For $I = \{0\}$ we get $\Gamma = \mathbb{R}$ and $K = \mathbb{R}[[\omega^{\mathbb{R}}]]$; note that K is closed under ∂_{BM}, but $\omega \in K$ and $\log \omega = \omega^{1/\omega} \notin K$.

For $I = \{0, -1\}$ we have $\Gamma = \mathbb{R} + \mathbb{R}\omega^{-1}$, so $\omega^\Gamma = \omega^{\mathbb{R}}(\log \omega)^{\mathbb{R}}$, and thus $K = \mathbb{R}[[\omega^{\mathbb{R}}(\log \omega)^{\mathbb{R}}]]$, which is again closed under ∂_{BM}.

Let $I = \{\alpha :\ \alpha \leqslant \varepsilon\}$. Then $\varepsilon = \omega^{\omega^\varepsilon} \in K$, but Lemma 2.10 gives $\log \varepsilon \notin K$, since $\varepsilon - 1 \notin I$ and so $\omega^{\varepsilon-1} \notin \Gamma$. Likewise we get $\partial_{\mathrm{BM}}(\varepsilon) \notin K$.

Lemma 3.3. *If $I = \{a :\ l(a) < \alpha\}$ or $I = \{a :\ l(a) \leqslant \alpha\}$, then $I \subseteq \Gamma \subseteq K$.*

Proof. Suppose $I = \{a :\ l(a) < \alpha\}$. (The case $I = \{a :\ l(a) \leqslant \alpha\}$ is handled in the same way.) Let $a \in I$. Then $a = \sum_x a_x \omega^x$, and if $x \in E(a)$, then $l(x) \leqslant l(\omega^x) \leqslant l(a) < \alpha$ by [5, Lemmas 3.4, 4.1, and 4.2], so $x \in I$. Thus $a \in \Gamma$. This proves $I \subseteq \Gamma$. Next, if $b \in \Gamma$, then $b = \sum_{x \in I} b_x \omega^x$, and so $b \in K$ in view of $I \subseteq \Gamma$. □

The next lemma will also be crucial:

Lemma 3.4. *Suppose $h(I) \subseteq \Gamma$. Then $\log K^{>} \subseteq K$ and for each $a \in K$ and term t of a we have: t and all terms of $\ell(t)$ lie in K.*

Proof. Let $a \in K^{>}$ have leading monomial $\mathfrak{m} = \omega^b$ with $b = \sum_{y \in I} b_y \omega^y$; to get $\log a \in K$, it is enough that $\log \mathfrak{m} \in K$; the latter holds because $\log \mathfrak{m} = \sum_y b_y \omega^{h(y)}$. This proves $\log K^{>} \subseteq K$.

Next, let $a \in K$ and let t be a term of a; we have to show that t and all terms of $\ell(t)$ lie in K. As $K \supseteq \mathbb{R}$ is initial, it does contain the term t of its element a. We have $t = r\omega^b$ with $r \in \mathbb{R}^\times$ and $b \in \Gamma$, so $b = \sum_{y \in I} b_y \omega^y$, and thus $\omega^b = \exp\left(\sum_{y \in I} b_y \omega^{h(y)}\right)$. Hence $\ell(t) = \ell(r\omega^b) = \sum_{y \in I} b_y \omega^{h(y)}$ and each of its terms $b_y \omega^{h(y)}$ lies obviously in K. □

Corollary 3.5. *If $h(I) \subseteq \Gamma$ and D is a pre-derivation with $D(K \cap \mathfrak{A}) \subseteq K$, then $\partial_D(K) \subseteq K$.*

Proof. Use the definition of ∂_D from Section 1, the fact that K is closed under infinite sums, and Lemma 3.4. □

Corollary 3.6. *Suppose $h(I) \subseteq \Gamma$. Then $\partial_{\mathrm{BM}}(K) \subseteq K$.*

Proof. Let $\lambda \in K \cap \mathfrak{A}$; by Corollary 3.5 we just need to get $\partial_{\mathfrak{A}}(\lambda) \in K$. Since K is closed under infinite products, it is enough for this to get $\log_n \lambda \in K$ for all n (which is the case by Lemma 3.4), and $\lambda_{-\alpha} \in K$ for all α such that $\lambda \preccurlyeq^K \lambda_{-\alpha}$. Given such α, take n with $\log_n \lambda < \lambda_{-\alpha}$. Then $\lambda_{-\alpha} \leqslant_s \lambda_{-(\alpha+1)} \leqslant_s \log_n \lambda \in K$ by Lemma 2.7, and so $\lambda_{-\alpha} \in K$ because K is initial. □

It can happen that $h(I) \not\subseteq \Gamma$ and that K is nevertheless closed under ∂_{BM}. The next lemma gives a useful criterion for that. To see why that lemma holds, consider a surreal derivation ∂, and note that from $\omega^{\omega^y} = \exp(\omega^{h(y)})$ we get

$$\partial\left(\omega^{\omega^y}\right) \ = \ \omega^{\omega^y} \cdot \partial(\omega^{h(y)}),$$

so for any monomial $\mathfrak{m} = \omega^b \in K$ we have $b = \sum_{y \in I} b_y \omega^y$, and thus

$$\mathfrak{m} \ = \ \exp\left(\sum_{y \in I} b_y \omega^{h(y)}\right), \qquad \partial(\mathfrak{m}) \ = \ \mathfrak{m} \cdot \sum_{y \in I} b_y \partial\left(\omega^{h(y)}\right).$$

This leads to:

Lemma 3.7. *Given a surreal derivation ∂, the following are equivalent:*

(1) *K is closed under ∂;*
(2) *$\partial(\omega^{\omega^y}) \in K$ for all $y \in I$;*
(3) *$\partial(\omega^{h(y)}) \in K$ for all $y \in I$.*

The surreal fields K_ε. Given the ε-number ε, we have the initial set $I := \mathbf{No}(\varepsilon)$, with the corresponding $\Gamma := \mathbb{R}[[\omega^I]]$ and $K := \mathbb{R}[[\omega^\Gamma]]$. In view of Lemmas 3.1 and 3.3 we have $h(I) \subseteq I \subseteq \Gamma$, so $\partial_{\mathrm{BM}}(K) \subseteq K$ by Corollary 3.6. Thus K is a spherically complete initial H-subfield of $\mathbf{No}$. However, I has no least element, so K is not grounded. We repair this by just augmenting I by $-\varepsilon$: set $I_\varepsilon := I \cup \{-\varepsilon\}$. Then I_ε is still initial, with least element $-\varepsilon$, and so we have the corresponding $\Gamma_\varepsilon := \mathbb{R}[[\omega^{I_\varepsilon}]]$ and $K_\varepsilon := \mathbb{R}[[\omega^{\Gamma_\varepsilon}]]$. To get $\partial_{\mathrm{BM}}(K_\varepsilon) \subseteq K_\varepsilon$ we note that $K \subseteq K_\varepsilon$, and so it suffices by Lemma 3.7 that $\partial_{\mathfrak{A}}(\omega^{\omega^{-\varepsilon}}) \in K_\varepsilon$. But $\omega^{\omega^{-\varepsilon}} = \log_\varepsilon \omega$, and

$$\partial_{\mathfrak{A}}(\log_\varepsilon \omega) \ = \ 1 \Big/ \prod_{\alpha<\varepsilon} \log_\alpha \omega,$$

which lies in K, and hence in K_ε. Thus K_ε is a grounded H-subfield of $\mathbf{No}$, and

$$\mathbf{No} \ = \ \bigcup_\varepsilon K_\varepsilon.$$

Note that Corollary 3.6 does not apply to I_ε, since $h(-\varepsilon) = \omega^{-(\varepsilon+1)} \notin \Gamma$; this is why we did the less direct construction via $I = \mathbf{No}(\varepsilon)$.

Since $\omega^{-\varepsilon}$ represents the smallest archimedean class of Γ_ε, we have

$$\max \Psi_{K_\varepsilon} \ = \ v\big((\omega^{\omega^{-\varepsilon}})^\dagger\big) \ = \ v\big((\log_\varepsilon \omega)^\dagger\big)$$

by the proof of Lemma 1.1. In view of $(\log_\varepsilon \omega)^\dagger = (\log_{\varepsilon+1} \omega)'$ and the remarks at the end of Section 1, the representation of $\mathbf{No}$ as an increasing union $\bigcup_\varepsilon K_\varepsilon$ of spherically complete grounded H-subfields now gives $\partial_{\mathrm{BM}}(\mathbf{No}) = \mathbf{No}$. (The proof of $\partial_{\mathrm{BM}}(\mathbf{No}) = \mathbf{No}$ in [3, Section 7] is different.) Thus by the results stated at the end of Section 1 we conclude that $\mathbf{No} \equiv \mathbb{T}$, as differential fields.

4. The Case of Restricted Length

A set $S \subseteq \mathbf{No}$ is said to be of *countable type* if $l(a)$ is countable for all $a \in S$, and all well-ordered subsets of S as well as all reverse well-ordered subsets of S are countable. (Note that $l(a)$ is countable for every $a \in \mathbf{No}(\omega_1)$, but that $\mathbf{No}(\omega_1)$ is not of countable type, since it has the set of countable ordinals as an uncountable well-ordered subset.)

Proposition 4.1. *Suppose the subset S of $\mathbf{No}$ is of countable type. Then the additive subgroup $\mathbb{R}[[\omega^S]]$ of $\mathbf{No}$ is also of countable type.*

Proof. The case $\alpha = 1$ of Esterle [8, Lemme 2.2] and the remarks following it yield that every well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable. Hence every reverse well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable as well. Let $a \in \mathbb{R}[[\omega^S]]$. Then $a = \sum_{s \in E(a)} a_s \omega^s$. Now $E(a) \subseteq S$ is countable, so the well-ordered set $-E(a)$ has order type $\mu < \omega_1$. Since ω_1 is regular, we have a countable ordinal ν such that $l(s) \leqslant \nu$ for all $s \in E(a)$. Then $l(\omega^s) \leqslant \omega^\nu$ for all $s \in E(a)$ by [5, Lemma 4.1], hence $l(a_s\omega^s) \leqslant \omega^{\nu+1}$ for all $s \in E(a)$ by [5, Proposition 3.6]. Thus

$$l(a) \ \leqslant \ \mu \cdot \omega^{\nu+1} \ < \ \omega_1,$$

by [9, Theorem 5.12], or [5, Lemma 4.2, (3)]. ☐

As an example, consider $S := \mathbf{No}(\omega)$, the set of of dyadic numbers. Then S is of countable type, and so $\mathbb{R}[[\omega^S]]$ is of countable type. Nevertheless, $l\big(\mathbb{R}[[\omega^S]]\big)$ is cofinal in ω_1: given any countable ordinal μ, take an order reversing injective map $\alpha \mapsto s_\alpha \colon \mu \to S$; then $a := \sum_\alpha \omega^{s_\alpha} \in \mathbb{R}[[\omega^S]]$ has $l(a) \geqslant \mu$, by [9, p. 63].

Let κ be any infinite cardinal. Esterle [8, Lemme 2.2] actually tells us for any set $S \subseteq \mathbf{No}$: if all well-ordered subsets and all reverse well-ordered subsets of S have size $\leqslant \kappa$, then this remains true for the set $\mathbb{R}[[\omega^S]] \subseteq \mathbf{No}$. The next cardinal κ^+ is regular, so the arguments in the proof of Proposition 4.1 go through to give the following, where we call $S \subseteq \mathbf{No}$ *of type* κ if $l(a) \leqslant \kappa$ for all $a \in S$ and all well-ordered subsets of S and all reverse well-ordered subsets of S have size $\leqslant \kappa$.

Corollary 4.2. *If $S \subseteq \mathbf{No}$ is of type κ, then so is $\mathbb{R}[[\omega^S]]$.*

Next we show that for countable μ the set $\mathbf{No}(\mu)$ is of countable type. Every element of $\mathbf{No}(\mu)$ has clearly countable length, for countable μ, and $\mathbf{No}(\mu)$ is closed under $x \mapsto -x$, so the assertion above reduces to:

Lemma 4.3. *Suppose the ordinal μ is countable. Then every well-ordered subset of $\mathbf{No}(\mu)$ is countable.*

This may remind the reader of the well-known property of the ordered set $\mathbb{R}$ that every well-ordered subset of $\mathbb{R}$ is countable. Here is a quick proof using that $\mathbb{R}$ has a countable dense subset $\mathbb{Q}$: given any embedding $\alpha \mapsto r_\alpha$ of an infinite cardinal κ into $\mathbb{R}$, pick for each $\alpha < \kappa$ a rational q_α such that $r_\alpha < q_\alpha < r_{\alpha+1}$; it follows that $\kappa = \aleph_0$. However, such a countable density argument cannot be used for ordered sets $\mathbf{No}(\mu)$ when μ is a countable limit ordinal $> \omega$:

Lemma 4.4. *Let μ be an infinite limit ordinal. Then the ordered set $\mathbf{No}(\mu)$ is dense without endpoints. If $\mu > \omega$, then there exists a collection of $2^{\aleph_0}$ pairwise disjoint open intervals in $\mathbf{No}(\mu)$, which has therefore no countable dense subset.*

Proof. The ordinals $\alpha < \mu$ are cofinal in this ordered set, and there is no largest such α. For $a < b$ in this ordered set, take $\alpha \leqslant l(a), l(b)$ such that $a|_\alpha = b|_\alpha$ and $a(\alpha) < b(\alpha)$. If $l(b) > \alpha$, then $b(\alpha) = +$, so $a < b- < b$. If $l(a) > \alpha$, then $a(\alpha) = -$, so $a < a+ < b$. Note that $b-, a+ \in \mathbf{No}(\mu)$, as μ is a limit ordinal,

Next, assume $\mu > \omega$. For each nondyadic $r \in \mathbb{R} \subseteq \mathbf{No}$, we have the surreals $r-$ and $r+$ of length $\omega+1$, and so we obtain the pairwise disjoint open intervals $(r-, r+)$ in $\mathbf{No}(\mu)$. $\square$

Proof of Lemma 4.3. For $a \in \mathbf{No}(\mu)$ we define $\widehat{a} \colon \mu \to \mathbb{R}$ by

$$\widehat{a}(\alpha) \;=\; \begin{cases} -1 & \text{if } a(\alpha) = -, \\ 0 & \text{if } a(\alpha) = 0, \\ 1 & \text{if } a(\alpha) = +, \end{cases}$$

For $S = \{\alpha :\ \alpha < \mu\}$ this yields an order-preserving injective map

$$a \mapsto \sum_{\alpha<\mu} \widehat{a}(\alpha)\omega^{-\alpha} \;:\; \mathbf{No}(\mu) \to \mathbb{R}[[\omega^S]].$$

It remains to appeal to Proposition 4.1. $\square$

Essentially the same argument yields the following generalization:

Corollary 4.5. *If κ is an infinite cardinal and μ is an ordinal of cardinality $\leqslant \kappa$, then each well-ordered subset of $\mathbf{No}(\mu)$ has cardinality $\leqslant \kappa$.*

Note that for a countable ε-number ε the initial set $I_\varepsilon = \mathbf{No}(\varepsilon) \cup \{-\varepsilon\}$ is of countable type by Lemma 4.3, and hence Γ_ε and K_ε are as well by Proposition 4.1. Taking the union over all such countable ε we obtain the set $\mathbf{No}(\omega_1)$ of all surreals of countable length as an increasing union of spherically complete grounded H-subfields K_ε of $\mathbf{No}$. As in Section 3 and using also the model completeness of $T^{\mathrm{nl}}_{\mathrm{small}} = \mathrm{Th}(\mathbb{T})$ this yields Theorem 2. The results above lead moreover to the following generalization:

Corollary 4.6. *Let κ be any uncountable cardinal. Then the subfield $\mathbf{No}(\kappa)$ of $\mathbf{No}$ is closed under ∂_{BM}, and $\mathbf{No}(\kappa) \prec \mathbf{No}$, as ordered differential fields.*

Proof. If κ is regular we can argue as for ω_1, using Corollaries 4.2 and 4.5 instead of Proposition 4.1 and Lemma 4.3. If κ is singular, use that it is the supremum of the uncountable regular cardinals below it. □

5. Constructing Embeddings

So far we have just worked inside $\mathbf{No}$ and established Theorem 2. In this section we turn to $\mathbb{T}$ and prove the embedding results: Theorems 1 and 3.

Embedding $\mathbb{T}$ into No. Given a Hahn field $\mathbb{R}[[G]]$ over $\mathbb{R}$ we define a map $F\colon \mathbb{R}[[G]] \to \mathbf{No}$ to be *strongly additive* if for every summable family (f_i) in $\mathbb{R}[[G]]$ the family $\big(F(f_i)\big)$ is summable in $\mathbf{No}$ and $F\left(\sum_i f_i\right) = \sum_i F(f_i)$. We refer to [2, Appendix A] for the construction of $\mathbb{T}$ as an exponential ordered field. In this construction $\mathbb{T}$ is a subfield of a Hahn field $\mathbb{R}[[G^{\mathrm{LE}}]]$: in fact, G^{LE} is a certain directed union of ordered subgroups $G_m{\downarrow_n}$, and $\mathbb{T}$ is the corresponding directed union of the Hahn fields $\mathbb{R}[[G_m{\downarrow_n}]]$. A map $F\colon \mathbb{T} \to \mathbf{No}$ is said to be *strongly additive* if its restriction to each $\mathbb{R}[[G_m{\downarrow_n}]]$ is strongly additive.

Proposition 5.1. *There is a unique strongly additive embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ of exponential ordered fields that is the identity on $\mathbb{R}$ and such that $\iota(x_{\mathbb{T}}) = \omega$.*

Proof. We use the notations from [2, Appendix A] except that the x there is $x_{\mathbb{T}}$ here. The construction of $\mathbb{T}$ there begins with the Hahn field $E_0 = \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]$, and there is clearly a (unique) strongly additive ordered field embedding $i_0\colon E_0 \to \mathbf{No}$ such that $i_0(r) = r$ and $i_0(x_{\mathbb{T}}^r) = \omega^r$ for all $r \in \mathbb{R}$. Moreover, $i_0(\mathrm{e}^b) = \exp(i_0(b))$ for all $b \in B_0$, and $\exp(i_0(a)) > i_0(E_0)$ for all $a \in A_0^>$. Assume inductively that we have an extension of i_0 to a strongly additive ordered field embedding $i_m\colon E_m = \mathbb{R}[[G_m]] \to \mathbf{No}$ such that $i_m(\mathrm{e}^b) = \exp(i_m(b))$ for all $b \in B_m$, and $\exp(i_m(a)) > i_m(E_m)$ for all $a \in A_m^>$. Then one checks easily that i_m extends (uniquely) to a strongly additive ordered field embedding $i_{m+1}\colon E_{m+1} \to \mathbf{No}$ such that $i_{m+1}(\mathrm{e}^b) = \exp(i_{m+1}(b))$ for all $b \in B_{m+1}$, and $\exp(i_{m+1}(a)) > i_{m+1}(E_{m+1})$ for all $a \in A_{m+1}^>$. Taking a union over all m we obtain an embedding

$$\iota_0 := \bigcup_m i_m : \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]^{\mathrm{E}} = \bigcup_m \mathbb{R}[[G_m]] \to \mathbf{No}$$

of ordered exponential fields. Replacing in the above $\ell_0 = x_{\mathbb{T}}$, G_m, ω, by $\ell_n = \log_n x_{\mathbb{T}}$, $G_m{\downarrow_n}$, $\log_n \omega$, respectively, we obtain likewise an embedding

$$\iota_n : \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}} = \bigcup_m \mathbb{R}[[G_m{\downarrow_n}]] \to \mathbf{No}$$

of ordered exponential fields with $\iota_n(\ell_n) = \log_n \omega$. Each ι_{n+1} extends ι_n, so we can take the union over all n to get an embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ as claimed. The uniqueness holds because the smallest subfield of $\mathbb{T}$ that contains $\mathbb{R}(x_{\mathbb{T}})$ and is closed under exponentiation, taking logarithms of positive elements, and summation of summable families is $\mathbb{T}$ itself. $\square$

Next we apply the model completeness of the theory of the exponential ordered field of real numbers (Wilkie [13]). By [6] and [5], respectively, the ordered exponential fields $\mathbb{T}$ and $\mathbf{No}$ are models of this theory, and so $\iota\colon \mathbb{T} \to \mathbf{No}$ is an elementary embedding of ordered exponential fields.

It is easy to check that $\iota\colon \mathbb{T} \to \mathbf{No}$ is also an embedding of ordered differential fields. In view of $\mathbb{T} \equiv \mathbf{No}$ (as differential fields), and the model completeness of $T^{\mathrm{nl}}_{\mathrm{small}}$ mentioned at the end of Section 1 we conclude that ι is an elementary embedding of ordered differential fields: Theorem 1.

Is ι an elementary embedding of *ordered differential exponential fields*? We don't know; this is related to the open problem from [2] to extend the model-theoretic results there about $\mathbb{T}$ as a differential field to $\mathbb{T}$ as a differential exponential field.

It follows easily from the construction of $\mathbb{T}$ and ι that all surreal derivations ∂ with $\partial(\omega) = 1$ agree on $\iota(\mathbb{T})$.

Proposition 5.2. *Here are some further properties of the map ι:*

(1) $\iota(G^{\mathrm{LE}}) = \mathfrak{M} \cap \iota(\mathbb{T})$*;*
(2) $\iota(\mathbb{T})$ *is truncation closed;*
(3) $\iota(\mathbb{T})$ *is of countable type; in particular,* $\iota(\mathbb{T}) \subseteq \mathbf{No}(\omega_1)$.

Proof. Induction on m gives $\iota(G_m) \subseteq \mathfrak{M}$, where we use at the inductive step that $G_{m+1} = \exp(A_m)G_m$ and $\iota(A_m) \subseteq \mathbb{J}$, the latter being a consequence of $\iota(G_m) \subseteq \mathfrak{M}$. Likewise, $\iota(G_m{\downarrow_n}) \subseteq \mathfrak{M}$ for all m, n, and thus $\iota(G^{\mathrm{LE}}) \subseteq \mathfrak{M}$. Since ι respects infinite sums of monomials, this yields (1), and (2) is then an immediate consequence using also that $\mathbb{T}$ is truncation closed in $\mathbb{R}[[G^{\mathrm{LE}}]]$. As to (3), using the results in Section 4 one shows by induction on m that $\iota(G_m)$, and likewise each $\iota(G_m{\downarrow_n})$, has countable type. Hence $\iota(G^{\mathrm{LE}})$ has countable type, and so does $\iota(\mathbb{T})$. $\square$

Question (Elliot Kaplan): can (2) be improved to $\iota(\mathbb{T})$ being initial?

Embedding H-fields into No. Let ε be an ε-number; for example, ε could be any uncountable cardinal. We recall from [5] that $\mathbf{No}(\varepsilon)$ is a real closed subfield of $\mathbf{No}$ containing $\mathbb{R}$. We consider $\mathbf{No}(\varepsilon)$ as a valued subfield of $\mathbf{No}$ with (divisible) ordered value group $v\big(\mathbf{No}(\varepsilon)^\times\big)$. We shall need an easy auxiliary result:

Lemma 5.3. *Let κ be a regular uncountable cardinal. Then the underlying ordered sets of $\mathbf{No}(\kappa)$ and $v\big(\mathbf{No}(\kappa)^\times\big)$ are κ-saturated.*

Proof. Let $A, B \subseteq \mathbf{No}(\kappa)$ have cardinality $< \kappa$, with $A < B$. The regularity of κ yields an ordinal $\alpha < \kappa$ such that $l(A \cup B) < \alpha$. By [9, Theorem 2.3] this gives a surreal a with $l(a) \leqslant \alpha$ such that $A < a < B$, and then $a \in \mathbf{No}(\kappa)$. Thus $\mathbf{No}(\kappa)$ is κ-saturated as an ordered set. Next, let $P, Q \subseteq \mathbf{No}(\kappa)^>$ have cardinality $< \kappa$, with $v(P) > v(Q)$. Set $A := \{np :\ n \geqslant 1,\ p \in P\}$ and $B := \{q/n :\ n \geqslant 1,\ q \in Q\}$. Then $A < B$, and so the above gives $a \in \mathbf{No}(\kappa)$ with $A < a < B$. Then $v(P) > v(a) > v(Q)$, showing that $v\big(\mathbf{No}(\kappa)^\times\big)$ is κ-saturated as an ordered set. $\square$

For Theorem 3 we need a sharpening of the model completeness of the theory T^{nl} of $\upomega$-free newtonian Liouville closed H-fields, namely, the quantifier elimination (QE) explained in [2, Introduction to Chapter 16]. The relevant first-order language for

QE has in addition to $\mathcal{L}$ extra unary predicate symbols $\mathrm{I}, \Lambda, \Omega$, to be interpreted in a model L of T^{nl} as sets $\mathrm{I}(L), \Lambda(L), \Omega(L) \subseteq L$ according to their defining axioms:

$$\begin{aligned} \mathrm{I}(a) &\iff a = y' \text{ for some } y \prec 1 \text{ in } L, \\ \Lambda(a) &\iff a = -y^{\dagger\dagger} \text{ for some } y \succ 1 \text{ in } L, \\ \Omega(a) &\iff 4y'' + ay = 0 \text{ for some } y \in L^{\times}. \end{aligned}$$

The sets $\mathrm{I}(L), \Lambda(L), \Omega(L) \subseteq L$ are convex; their role with respect to QE is like that of the set of squares in a real closed field. For more on this, see [2, Introduction]. A $\Lambda\Omega$-*field* is a substructure $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ of such an expanded model $(L, \dots)$ of T^{nl} for which K is an H-subfield of L. This notion of a $\Lambda\Omega$-field is studied in detail in [2, Section 16.3], from which we take in particular the fact that any ω-free H-field K has a unique expansion to a $\Lambda\Omega$-field $\boldsymbol{K} = (K, I, \Lambda, \Omega)$. The proof below assumes familiarity with several other results from [2, Section 16.3].

Proof of Theorem 3. Let $\mathbf{No}_{\Lambda\Omega}$ be the expansion of $\mathbf{No}$ to a $\Lambda\Omega$-field, and let K be any H-field with small derivation and constant field $\mathbb{R}$. In order to embed K over $\mathbb{R}$ into $\mathbf{No}$, we first expand K to a $\Lambda\Omega$-field $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ with $1 \notin I$; this can be done in at least one way, and at most two ways, and $1 \notin I$ guarantees that all $\Lambda\Omega$-field extensions of $\boldsymbol{K}$ have small derivation. We claim that $\boldsymbol{K}$ can be embedded into $\mathbf{No}_{\Lambda\Omega}$. The ordered field $\mathbb{R}$ with the trivial derivation is an H-field and expands to the $\Lambda\Omega$-field $\boldsymbol{R} := \big(\mathbb{R}, \{0\}, (-\infty, 0], (-\infty, 0]\big)$. The inclusion of $\mathbb{R}$ into K and into $\mathbf{No}$ are embeddings of $\boldsymbol{R}$ into $\boldsymbol{K}$ and $\mathbf{No}_{\Lambda\Omega}$, respectively. By taking $\boldsymbol{E} := \boldsymbol{R}$, our claim reduces therefore to proving the following more general statement:

Claim. Let $\boldsymbol{E} \subseteq \boldsymbol{K}$ be an extension of $\Lambda\Omega$-fields with $\mathbb{R}$ as their common constant field, and let $i\colon \boldsymbol{E} \to \mathbf{No}_{\Lambda\Omega}$ be an embedding of $\Lambda\Omega$-fields that is the identity on $\mathbb{R}$. Then i extends to an embedding $\boldsymbol{K} \to \mathbf{No}_{\Lambda\Omega}$ of $\Lambda\Omega$-fields.

To prove this we first extend $\boldsymbol{K}$ to make it ω-free, newtonian, and Liouville closed; by [2, 16.4.1 and 14.5.10] this can be done without changing its constant field. Next we apply [2, 16.4.1] again, but this time to $\boldsymbol{E}$, to arrange that $\boldsymbol{E}$ is ω-free. Take a regular uncountable cardinal $\kappa > \operatorname{card}(K)$ such that $i(E) \subseteq \mathbf{No}(\kappa)$, where E is the underlying set of $\boldsymbol{E}$. By Corollary 4.6 we have $\mathbf{No}(\kappa) \prec \mathbf{No}$. In view of Lemma 5.3 and [2, 16.2.3] we can then extend i to an embedding $K \to \mathbf{No}(\kappa)$. $\square$

Final remarks. Suppose the H-field K has small derivation and constant field $\mathbb{R}$. Then Theorem 3 yields an embedding $i\colon K \to \mathbf{No}$ over $\mathbb{R}$. Under some reasonable further conditions, like K being ω-free and newtonian, can we take i such that $i(K)$ is truncation closed, or even initial? The interest of such a result would depend on how canonical the derivation ∂_{BM} is deemed to be. As already mentioned at the end of the introduction, we doubt that ∂_{BM} is optimal: the condition on pre-derivations to take values in $\mathbb{R}^{>}\mathfrak{M}$ seems too narrow. But even with this restriction one can construct pre-derivations $D \neq \partial_{\mathfrak{L}}$ such that Theorems 1 and 3 go through for $\mathbf{No}$ equipped with ∂_D instead of with ∂_{BM}, with only minor changes in the proofs.

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